Algebraic geometry 1

Exercise Sheet 8

PD Dr. Maksim Zhykhovich Winter Semester 2025, 12.12.2025

Exercise 1. Let $Y \subset \mathbb{P}^m$ be a quasi-projective algebraic set and let $\varphi : \mathbb{P}^n \to Y$ be a morphism. Show that there exists homogeneous polynomials $F_0, \ldots, F_m \in K[X_0, X_1, \ldots, X_n]$ of the same degree and with no common zero on $\mathbb{P}^n$ such that

$$\varphi(x) = [F_0(x), \ldots, F_m(x)] \text{ for all } x \in \mathbb{P}^n .$$

Remark: By definition of a morphism the above condition is satisfied locally.

Exercise 2. Let K be an algebraically closed field of characteristic different from two. Let $f = \sum_{ij} a_{ij} X_i X_j \in K[X_0, X_1, X_2, X_3]$ be a quadratic homogeneous polynomial with $a_{ij} = a_{ji} \in K$.
Assume that f is non-degenerate, that is the corresponding symmetric matrix $A := (a_{ij}) \in M_4(K)$ is invertible, and consider a projective quadric $V^p(f) \subset \mathbb{P}^3$
(1) Show that $V^p(f) \simeq V^p(X_0^2 + X_1^2 + X_2^2 + X_3^2)$.
Hint: Use results from linear algebra and perform a linear transformation of variables.
(2) Show that $V^p(X_0^2 + X_1^2 + X_2^2 + X_3^2) \simeq V^p(X_0X_1 - X_2X_3)$.
(3) Deduce that $V^p(f) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for any non-degenerate f.

Exercise 3. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$ be a morphism. Show that the image of φ is closed.
Hint: Extend φ to a morphism $\widetilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^n$ and use Theorem 5.19 (we will prove this theorem next week).

Exercise 4. Consider the morphism $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$.
(1) Show that the image of φ is not closed in $\mathbb{A}^2$.
(2) Try to repeat the proof of Exercise 3 and see where the proof fails.